JOURNAL OF THERMOPHYSICS AND HEAT TRANSFER
Vol. 25, No. 3, July–September 2011

Technical Notes

Analytical Investigation of Heat Transfer in Couette–Poiseuille Flow Through Porous Medium

Orhan Aydın* and Mete Avcı†
Karadeniz Technical University, 61080 Trabzon, Turkey

DOI: 10.2514/1.45152

Nomenclature

Br	=	Brinkman number, Eq. (20)
c_p	=	specific heat at constant pressure, $\mathrm{Jkg^{-1}K^{-1}}$
Da	=	Darcy number, K/H^2
D_h	=	hydraulic diameter, $2H$
G	=	applied pressure gradient, $\mathrm{Nm^{-3}}$
H	=	channel width, m
K	=	permeability, $\mathrm{m^2}$
k	=	fluid thermal conductivity, $\mathrm{Wm^{-1}K^{-1}}$
M	=	μ_{eff}/μ
Nu	=	Nusselt number, Eq. (35)
q''	=	wall heat flux, $\mathrm{Wm^{-2}}$
S	=	$(MDa)^{-1/2}$
T^*	=	temperature, K
T_m^*	=	mean or bulk temperature, K
T_w^*	=	wall temperature, K
U	=	dimensionless mean velocity, Eq. (9)
U^*	=	mean velocity, Eq. (8), $\mathrm{ms^{-1}}$
u	=	$\mu_{\mathrm{eff}}u^*/GH^2$
$\hat{u}$	=	u^*/U^* or u/U
u^*	=	filtration velocity, $\mathrm{ms^{-1}}$
v	=	$\mu_{\mathrm{eff}}v^*/GH^2$
$\hat{v}$	=	v^*/U^* or v/U
v^*	=	axial velocity of the moving plate, $\mathrm{ms^{-1}}$
Y	=	y^*/H
y^*	=	vertical coordinate, m
z^*	=	axial coordinate, m
θ	=	$(T^*-T_w^*)/(q''H/k)$
θ_m	=	$(T_m^*-T_w^*)/(q''H/k)$
μ	=	fluid viscosity, $\mathrm{kgm^{-1}s^{-1}}$
μ_{eff}	=	effective viscosity, $\mathrm{kgm^{-1}s^{-1}}$
ρ	=	fluid density, $\mathrm{kgm^{-3}}$
Φ	=	viscous dissipation term

I. Introduction

VISCOUS dissipation plays a role like an internal heat source in the energy transfer, which, in the following, affects temperature distributions and heat transfer rates. This heat source is caused by the shearing of fluid layers. For a clear fluid, this effect has been studied in detail in the existing literature [1–3]. However, for the case of a porous medium, there are not many studies.

The flow and heat transfer in the combined form of the Couette and Poiseuille flows between parallel plates is important in many materials processing applications such as extrusion, metal forming, continuous casting, wire and glass fiber drawing, etc. Despite these broad-application fields, these flows have received less research interest than either the Couette flow only or the Poiseuille flow only. Aydın and Avcı [4] reviewed the existing literature in the study on the Couette–Poiseuille flow and studied the effect of the viscous dissipation for this problem.

The purpose of the present study is to analytically investigate the effect of viscous dissipation on steady-state laminar heat transfer in a Couette–Poiseuille flow of a porous medium between plane-parallel plates with a simultaneous pressure gradient and the axial movement of the upper plate. The effects of Brinkman number, Darcy number, and the upper-plate velocity on the Nusselt number is obtained for two different configurations of the thermal boundary conditions.

II. Analysis

Consider steady, hydrodynamically, and thermally fully developed laminar flow of an incompressible fluid between two parallel plates filled with a saturated porous medium (Fig. 1). The thermal conductivity and the thermal diffusivity of the fluid are considered to be independent of temperature. The upper plate is assumed to move at a constant velocity, whereas the lower one is stationary. The axial heat conduction in the fluid and in the wall is neglected.

The Brinkman momentum equation in the z^* direction is described as [5]

$$\mu_{\mathrm{eff}}\frac{\mathrm{d}^2u^*}{\mathrm{d}y^{*2}}-\frac{\mu}{K}u^*+G=0 \tag{1}$$

where μ_{eff} is the effective viscosity, μ is the fluid viscosity, K is the permeability, and G is the applied pressure gradient.

Using the following dimensionless parameters,

$$Y=\frac{y^*}{H},\qquad u=\frac{\mu_{\mathrm{eff}}u^*}{GH^2},\qquad v=\frac{\mu_{\mathrm{eff}}v^*}{GH^2}$$
$$M=\frac{\mu_{\mathrm{eff}}}{\mu},\qquad Da=\frac{K}{H^2} \tag{2}$$

the dimensionless form of Eq. (1) is written as

$$\frac{\mathrm{d}^2u}{\mathrm{d}Y^2}-S^2u+1=0 \tag{3}$$

Under the following boundary conditions,

$$Y=0,\qquad u=0,\qquad Y=1,\qquad u=v \tag{4}$$

Equation (3) is solved to give the dimensionless velocity distributions as

$$u=\frac{1}{S^2}+b_1e^{SY}+b_2e^{-SY} \tag{5}$$

where

$$S=(1/MDa)^{1/2} \tag{6}$$

In Eq. (5), b_1 and b_2 are the constants that are given, respectively, as

$$b_1=\frac{e^{-S}-1+vS^2}{S^2(e^S-e^{-S})},\qquad b_2=\frac{1-e^S-vS^2}{S^2(e^S-e^{-S})} \tag{7}$$

The mean velocity U^* is defined as

Received 28 April 2009; revision received 4 October 2010; accepted for publication 4 October 2010.

*Professor, Department of Mechanical Engineering; oaydin@ktu.edu.tr.
†Assistant Professor, Department of Mechanical Engineering; mavci@ktu.edu.tr.

$$U^* = \frac{1}{H}\int_0^H u^* \, dy^* \tag{8}$$

or in dimensionless form, as

$$U = \int_0^1 u \, dY \tag{9}$$

Integrating this equation gives

$$U = \frac{1}{S^2} + \frac{b_1}{S}(e^S - 1) - \frac{b_2}{S}(e^{-S} - 1) \tag{10}$$

Substituting Eq. (7) into Eq. (10) gives

$$U = \frac{1 - 2b_3}{S^2(1 - b_3\hat{v})} \tag{11}$$

where

$$b_3 = -\frac{2 - e^S - e^{-S}}{S(e^S - e^{-S})}, \qquad \hat{v} = v/U \tag{12}$$

After performing necessary substitutions, the dimensionless velocity $\hat{u}$ can be obtained as

$$\hat{u} = \frac{u^*}{U^*} = \frac{u}{U} = \frac{(1/S^2 + b_1 e^{SY} + b_2 e^{-SY})(1 - b_3\hat{v})(S^2)}{(1 - 2b_3)} \tag{13}$$

The conservation of energy including the effect of the viscous dissipation can be written as follows [5]:

$$\rho c_p u^* \frac{\partial T^*}{\partial z^*} = k\frac{\partial^2 T^*}{\partial y^{*2}} + \Phi \tag{14}$$

where the second term in the right-hand side is the viscous dissipation term.

Following the model proposed by Al-Hadhrami et al. [6,7], the viscous dissipation term expressed as

$$\Phi = \frac{\mu u^{*2}}{K} + \mu_{\text{eff}}\left(\frac{du^*}{\partial y^*}\right)^2 \tag{15}$$

which is compatible with an expression derived from the Navier–Stokes equation for a fluid clear of solid material, in the large Darcy number.

For the uniform-heat-flux case, the first law of thermodynamics results in the following relationship [6]:

$$\frac{\partial T^*}{\partial z^*} = \frac{dT_w^*}{dz^*} = \frac{dT_m^*}{dz^*} = \text{const} \tag{16}$$

In the above equation, T_w implies the hot wall temperature according to the thermal orientation of the channel (case A or case B).

Introduction of the following nondimensional temperature

$$\theta = \frac{T^* - T_w^*}{q''H/k} \tag{17}$$

modifies Eq. (14) into the following dimensionless form:

$$\frac{d^2\theta}{dY^2} = \left(\frac{\rho c_p H U^*}{q''}\frac{dT_w^*}{dz^*}\right)\hat{u} - \frac{2Br}{Da}\left(\hat{u}^2 + \frac{1}{S^2}\left(\frac{d\hat{u}}{dY}\right)^2\right) \tag{18}$$

or

$$\frac{d^2\theta}{dY^2} = b_4\hat{u} - b_5\left(\hat{u}^2 + \frac{1}{S^2}\left(\frac{d\hat{u}}{dY}\right)^2\right) \tag{19}$$

where b_4 is a constant unknown obtained by using thermal boundary conditions, b_5 is a group parameter, and Br is the Brinkman number given, respectively, as

$$b_4 = \frac{\rho c_p H U^*}{q'' U}\frac{dT_w^*}{dz^*} = \text{const}, \qquad b_5 = 2Br/Da, \qquad Br = \frac{\mu U^{*2}}{2q''H} \tag{20}$$

Integrating Eq. (19) twice, one obtains the general solution of energy equation as

$$\theta(Y) = -\frac{b_2^2 b_5 e^{-2SY}}{2S^2} - \frac{b_1^2 b_5 e^{2SY}}{2S^2} - \frac{b_2 e^{-SY}(2b_5 - b_4 S^2)}{S^4} - \frac{b_1 e^{SY}(2b_5 - b_4 S^2)}{S^4} + \frac{(b_4 S^2 - b_5)Y^2}{2S^4} + b_6 Y + b_7 \tag{21}$$

where b_6 and b_7 are the integration constants that can be found by using the corresponding thermal boundary conditions case A and case B, respectively.

Two different forms of the thermal boundary conditions are applied according to Aydın and Avcı [4], which are shown in Fig. 1. In the following, we treat these two different cases separately.

A. Case A

In this case (Fig. 1a), the thermal boundary conditions are as in the following:

$$T^* = T_w^*, \qquad k\frac{\partial T^*}{\partial y^*}\bigg|_{y^*=H} = q'' \quad \text{at } y^* = H, \qquad k\frac{\partial T^*}{\partial y^*}\bigg|_{y^*=0} = 0 \quad \text{at } y^* = 0 \tag{22}$$

Introducing the dimensionless temperature, Eq. (17), the thermal boundary conditions are written as

$$\theta = 0, \qquad \frac{\partial\theta}{\partial Y}\bigg|_{Y=1} = 1 \quad \text{at } Y = 1, \qquad \frac{\partial\theta}{\partial Y}\bigg|_{Y=0} = 0 \quad \text{at } Y = 0 \tag{23}$$

Using these boundary conditions, the constant unknown b_4 and the integration constants b_6 and b_7 for Eq. (21) are obtained, respectively, as

$$b_4 = \left(-1 + \frac{b_2^2 b_5(e^{-2S} - 1)}{S} - \frac{b_1^2 b_5(e^{2S} - 1)}{S} + \frac{2b_2 b_5(e^{-S} - 1)}{S^3} - \frac{2b_1 b_5(e^S - 1)}{S^3} - \frac{b_5}{S^4}\right) \Big/ \left(\frac{b_2(e^{-S} - 1)}{S} - \frac{b_1(e^S - 1)}{S} - \frac{1}{S^2}\right) \tag{24}$$

$$b_6 = \frac{b_5(b_1^2 - b_2^2)}{S} - \frac{(b_1 - b_2)(b_4 S^2 - 2b_5)}{S^3} \tag{25}$$

$$b_7 = \frac{b_5(b_2^2 e^{-2S} + b_1^2 e^{2S})}{2S^2} + \frac{(2b_5 - b_4 S^2)(b_2 e^{-S} + b_1 e^S)}{S^4} + \frac{b_5 - b_4 S^2}{2S^4} - b_6 \tag{26}$$

B. Case B

The thermal boundary conditions for case B (Fig. 1b) are as in the following,

$$k\frac{\partial T^*}{\partial y^*}\bigg|_{y^*=H} = 0 \quad \text{at } y^* = H, \qquad T^* = T_w^*, \qquad -k\frac{\partial T^*}{\partial y*}\bigg|_{y^*=0} = q'' \quad \text{at } y^* = 0 \tag{27}$$

or in dimensionless form, as

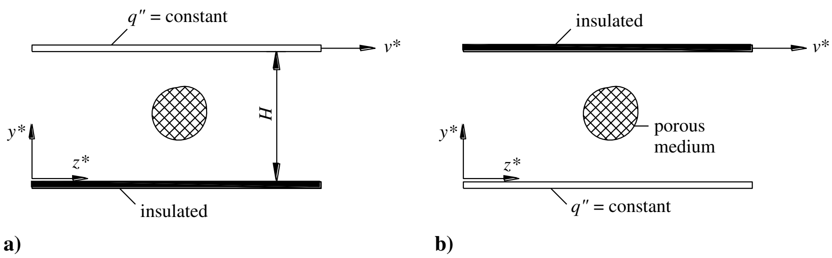

Fig. 1 Schematic diagram of the flow domain: a) case A and b) case B.

$$\left.\frac{\partial\theta}{\partial Y}\right|_{Y=1}=0 \quad \text{at } Y=1, \qquad \theta=0, \qquad \left.\frac{\partial\theta}{\partial Y}\right|_{Y=0}=-1 \quad \text{at } Y=0 \tag{28}$$

Similarly, using these boundary conditions, the integration constants b_6 and b_7 for Eq. (21) are obtained, respectively, as

$$b_6=-1+\frac{b_5(b_1^2-b_2^2)}{S}-\frac{(b_1-b_2)(b_4S^2-2b_5)}{S^3} \tag{29}$$

$$b_7=\frac{b_5(b_2^2+b_1^2)}{2S^2}+\frac{(2b_5-b_4S^2)(b_2+b_1)}{S^4} \tag{30}$$

Note that the constant unknown b_4 is the same for case A and case B.

In fully developed flow, it is usual to use the mean fluid temperature T_m^*, rather than the centerline temperature when defining the Nusselt number. This mean or bulk temperature is given by

$$T_m^*=\frac{\int_0^H u^*T^*\,\mathrm{d}y^*}{\int_0^H u^*\,\mathrm{d}y^*} \tag{31}$$

or in dimensionless form, as

$$\theta_m=\frac{T_m^*-T_w^*}{q''H/k}=\frac{\int_0^1 u\theta\,\mathrm{d}Y}{U} \tag{32}$$

Substituting Eq. (10) and Eq. (21) into Eq. (32), the dimensionless mean temperature is obtained as

$$\begin{aligned}\theta_m=\Bigg(&-\frac{b_7(b_1(1-e^S)+b_2(1-e^{-S}))}{S}-\frac{b_6(b_2e^{-S}-b_1e^S)}{S}\\&+\frac{b_4(b_1(1-e^S)+b_2(1-e^{-S})+1/2)}{S^2}+\frac{2b_4b_1b_2}{S^2}+\frac{b_7}{S^2}\\&+\frac{b_1b_2b_5(b_1(1-e^S)-b_2(1-e^{-S}))}{2S^3}\\&-\frac{b_4(b_2^2e^{-2S}-b_1^2e^{2S}+b_2e^{-S}-b_1e^S)}{2S^3}-\frac{b_4(b_1^2-b_2^2)}{2S^3}\\&+\frac{b_5(b_1^3(1-e^{3S})-b_2^3(1-e^{-3S}))}{6S^3}\\&+\frac{b_4(1/6-b_2e^{-S}-b_1e^S)}{S^4}-\frac{4b_1b_2b_5}{S^4}\\&+\frac{(b_5/2-2b_4)(b_2e^{-S}-b_1e^S)}{S^5}-\frac{2b_4(b_1-b_2)}{S^5}\\&+\frac{5b_5(b_1^2(1-e^{2S})-b_2^2(1-e^{-2S}))}{4S^5}-\frac{b_5(1/6-b_2e^{-S}-b_1e^S)}{S^6}\\&+\frac{3b_5(b_1(1-e^S)-b_2(1-e^{-S}))}{S^7}\Bigg)\Big/U\end{aligned} \tag{33}$$

The forced convective heat transfer coefficient is given as follows:

$$h=\frac{q''}{T_w^*-T_m^*} \tag{34}$$

which is obtained from Nusselt number that is defined as

$$Nu=\frac{q''D_h}{(T_w^*-T_m^*)k}=-\frac{2}{\theta_m} \tag{35}$$

where D_h is the hydraulic diameter of the cross section of the channel, $D_h=2H$.

III. Results and Discussion

Here, we study the Couette–Poiseuille flow in a saturated porous medium between two plane-parallel plates with a simultaneous pressure gradient and the axial movement of the upper plate. As stated earlier, the problem is steady, laminar, and hydrodynamically and thermally fully developed. Three different geometrical orientations of the upper plate are considered: The upper plate is 1) stationary 2) moving in the positive z direction, and 3) moving the

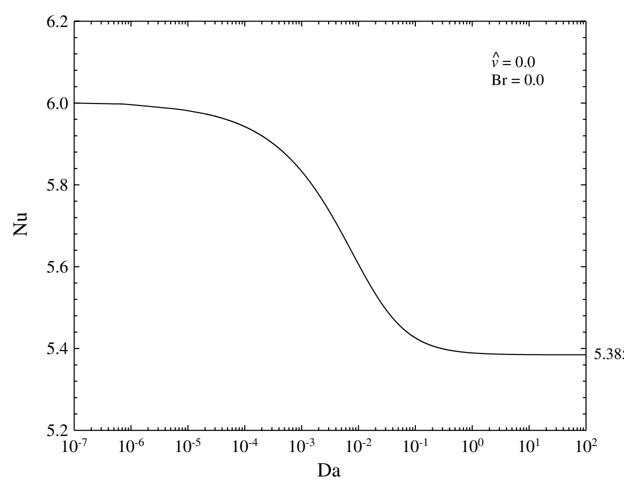

Fig. 2 Variations of the Nusselt number with the Darcy number at $Br = 0.0$ and $\hat{v} = 0$.

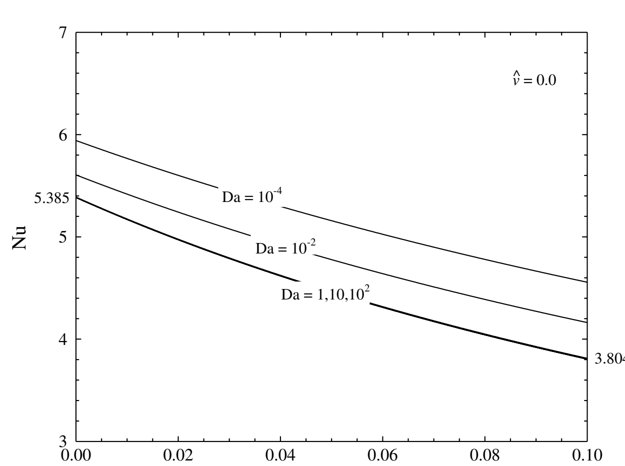

Fig. 3 Variations of the Nusselt number with the Brinkman number at different values of Darcy number for $\hat{v} = 0$.

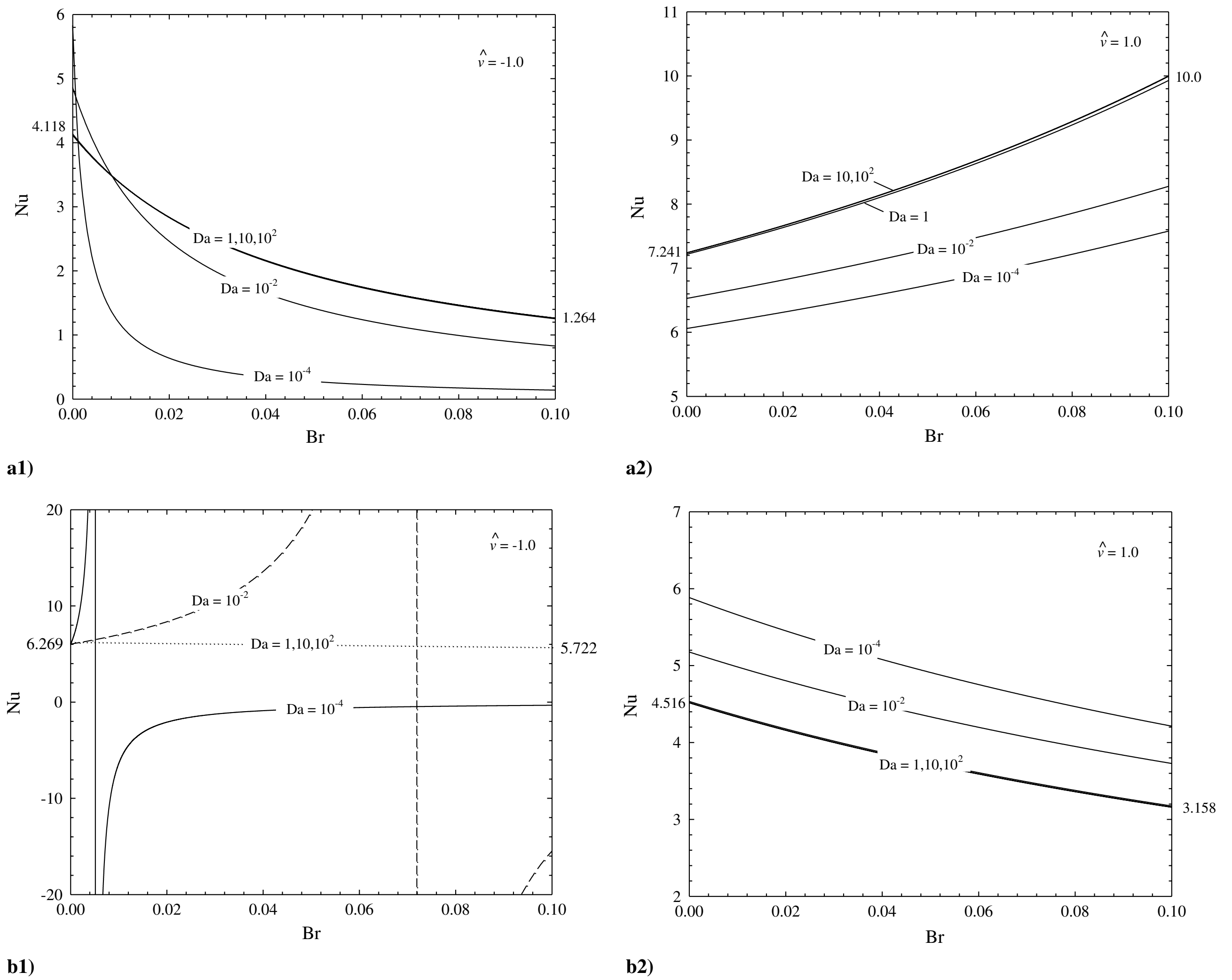

Fig. 4 Variations of the Nusselt number with the Brinkman number at different values of Darcy number for $\hat{v} = 1.0$ and -1.0: a) case A and b) case B.

negative z direction. For the sake of brevity and without loss of generality, it is assumed that $\mu_{\text{eff}} = \mu$, leading to $M = 1$, in the presentations of results.

Figure 2 illustrates the variation of Nusselt number with the Darcy number for the case without viscous dissipation effect. For the values of Darcy number lower than 10^{-4}, Nusselt number nearly stays constant at the slug flow value, $Nu = 6$, then it decreases sharply and finally it approaches to the clear fluid limit, $Nu = 5.385$ for $Da > 1$.

For the nonmoving-upper-wall case (the Poiseuille flow), the effect of the viscous dissipation on Nusselt number is shown in Fig. 3. As seen, Nusselt number decreases with an increase in Brinkman number. Increasing viscous dissipation increases both the wall temperature and the bulk fluid temperature. This increase is felt more in the wall due to high shear stress near the wall. It is clear from Eq. (35) that an increased value of the wall and mean temperature difference, $T_w^* - T_m^*$ will decrease Nusselt number.

Similarly, Figs. 4a and 4b illustrate the variation of the Nusselt number with the Brinkman number for different dimensionless relative velocity of the upper plate and Darcy number at cases A and B, respectively. For case A, an increase in Brinkman number decreases Nusselt number for the movement of the upper plate in the negative direction ($\hat{v} = -1$), while in the positive direction ($\hat{v} = 1$), it increases it. This is due to increasing temperature differences between the wall and the bulk fluid, as discussed above. For case B, similar behaviors are observed. However, for the movement of the upper plate in the negative direction ($\hat{v} = 1$), singularities are obtained at $Da = 10^{-2}$ and 10^{-4} due to the high shear rate near the wall. For $Da = 10^{-4}$, with the increasing value of Brinkman number, Nusselt number increases in the range of $0 < Br < 5.2 \times 10^{-3}$. This is because the temperature difference that drives the heat transfer decreases. At $Br = 5.2 \times 10^{-3}$, the heat supplied by the wall into the fluid is balanced with the internal heat generation due to the viscous heating. For $Br > 5.2 \times 10^{-3}$, the internally generated heat overcomes the heat supplied by the wall. When $Br \to 0.1$, the Nusselt number reaches an asymptotic value.

IV. Conclusions

The Couette–Poiseuille flow in a saturated porous medium between two plane-parallel plates with a simultaneous pressure gradient and axial movement of the upper plate was investigated analytically. The effect of the viscous dissipation was found to affect temperature profiles and heat transfer rates. The Nusselt numbers were determined for various values of Br, Da, and $\hat{v}$. It was disclosed that hydrodynamical and thermal behaviors of the porous medium approached the slug flow behaviors for the lower values of Da ($\leq 10^{-4}$), while the clear fluid behavior was observed for $Da \geq 1$.

References

[1] Aydın, O., "Effects of Viscous Dissipation on the Heat Transfer in a Forced Pipe Flow Part 1. Both Hydrodynamically and Thermally Fully Developed Flow," *Energy Conversion and Management*, Vol. 46, No. 5, 2005, pp. 757–769.
doi:10.1016/j.enconman.2004.05.004

[2] Aydın, O., "Effects of Viscous Dissipation on the Heat Transfer in a Forced Pipe Flow Part 2. Thermally Developing Flow," *Energy Conversion and Management*, Vol. 46, Nos. 18–19, 2005, pp. 3091–3102.

doi:10.1016/j.enconman.2005.03.011

[3] Aydın, O., and Avcı, M., "Viscous-Dissipation Effects on the Heat Transfer in a Poiseuille Flow," *Applied Energy*, Vol. 83, No. 5, 2006, pp. 495–512.
doi:10.1016/j.apenergy.2005.03.003

[4] Aydın, O., and Avcı, M., "Laminar Forced Convection with Viscous Dissipation in a Couette–Poiseuille Flow Between Parallel Plates," *Applied Energy*, Vol. 83, No. 8, 2006, pp. 856–867.
doi:10.1016/j.apenergy.2005.08.005

[5] Nield, D. A., Kuznetsov, A. V., and Xiong, M., "Thermally Developing Forced Convection in a Porous Medium: Parallel Plate Channel or Circular Tube with Isothermal Walls," *Journal of Porous Media*, Vol. 7, No. 1, 2004, pp. 19–27.
doi:10.1615/JPorMedia.v7.i1.30

[6] Al-Hadhrami, A. K., Elliott, L., and Ingham, D. B., "Combined Free and Forced Convection in Vertical Channels of Porous Media," *Transport in Porous Media*, Vol. 49, No. 3, 2002, pp. 265–289.
doi:10.1023/A:1016290505000

[7] Al-Hadhrami, A. K., Elliott, L., and Ingham, D. B., "A New Model for Viscous Dissipation in Porous Media Across a Range of Permeability Values," *Transport in Porous Media*, Vol. 53, No. 1, 2003, pp. 117–122.
doi:10.1023/A:1023557332542